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Cherenkov Radiation Emission in uniaxial optical materials

A. Delbart[a], J. Derré, and R. Chipaux[b]

CEA, DSM/DAPNIA, CE-Saclay, 91191 Gif sur Yvette cedex, France

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Abstract. We report a theoretical study of the Cherenkov radiation emission in uniaxial optical materials. The formalism is based on the previous work of Muzicar (1961) whose results in terms of energetic properties of the emitted waves are corrected. This formalism is used to predict the Cherenkov radiation emission in a strongly birefringent sodium nitrate crystal ($NaNO_3$).

PACS. 41.60.Bq Cherenkov radiation – 42.25.Lc Birefringence

1 Introduction

Cherenkov radiation in an isotropic medium is well understood and used in high energy physics detectors. It was discovered in 1934 by Cherenkov [1] who observed that pure sulfuric acid in a platinum crucible close to a radium source emitted a weak blue radiation. All the following observations revealed that a charged particle moving in a transparent and isotropic medium of index of refraction n, with a velocity $V = c\beta$ larger than the phase velocity of light in that medium, emits this new radiation in a well-defined direction which only depends on the product βn. The radiation spectrum was found to be continuous and concentrated mainly in the blue-violet part of the spectrum. In 1937, Tamm and Frank [2] developped a theory on the basis of classical electrodynamics which completely explained the properties of this radiation.

A first theoretical study of the Cherenkov radiation in anisotropic media was due to Ginzburg [3] in 1940. A complete theory was then obtained in 1956 by Pafomov who extensively sudied the Cherenkov radiation. He investigated the case of the Cherenkov emission in anisotropic ferrites [4] and transposed his results for anisotropic dielectrics. He showed that, in the general case of a fast charged particle moving in an arbitrary direction in an uniaxial optical material, there should exist two noncircular conical surfaces, which correspond to the ordinary and extraordinary waves. The intensity distribution over these surfaces should not be uniform owing to the polarization features of the radiations.

Zrelov experimentally confirmed in 1964 [5] the geometrical properties of the cones in the two special cases of a particle travelling along the optical axis of the material or perpendicularly to it. As far as we know, the only experimental verification of the energetic properties of the radiation was carried out in 1965 by Gföller [6], who studied the Cherenkov radiation emitted by a ^{32}P 1.71 MeV β source in a strongly anisotropic $NaNO_3$ plate. He found 15% less total emitted intensity for a particle moving perpendicularly to the optical axis than for a particle moving along this axis. This observation is well explained in Pafomov's theory. But we found that the corresponding formulæ, when applied to an isotropic medium, give a number of emitted photons which erroneously depends on the choice of the reference axis.

Another approach was due to Muzicar [7] and led to the same conclusions concerning the geometrical properties of the two cones and the intensity of the ordinary wave. But his conclusions are different from those of Pafomov regarding the intensity distribution of the extraordinary wave. Moreover, he found the same total number of photons whatever the angle between the particle motion and the optical axis, not in agreement with the measurement of Gföller.

We present the complete properties of these Cherenkov cones the same way as Muzicar did. The new result concerns the number of extraordinary photons, which is now in agreement with Gföller and successfully applied to the isotropic case.

2 Basic formula and notations

Let a point charged particle move in an uniaxial material with a constant velocity $v = c\beta$ (c is the velocity of light in vacuum), and in a direction defined by a unit vector $\mathbf{r}$. We choose a coordinate system $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ with $\mathbf{x}_1$ along the optical axis $\mathbf{c}$ and $\mathbf{x}_2$ in the $(\mathbf{r}, \mathbf{c})$ plane (see Fig. 1). In this system, $\mathbf{r}$ is defined by $r_1 = \cos(\chi)$, $r_2 = \sin(\chi)$ and $r_3 = 0$. The dielectric tensor is diagonal with $\epsilon_{11} = \epsilon_\mathrm{e} = n_\mathrm{e}^2$ and $\epsilon_{22} = \epsilon_{33} = \epsilon_\mathrm{o} = n_\mathrm{o}^2$, where n_o and n_e are the ordinary and extraordinary refractive indices. The two eigenstates of propagation [8], the ordinary and extraordinary waves,

[a] e-mail: delbart@hep.saclay.cea.fr
[b] e-mail: chipaux@hep.saclay.cea.fr

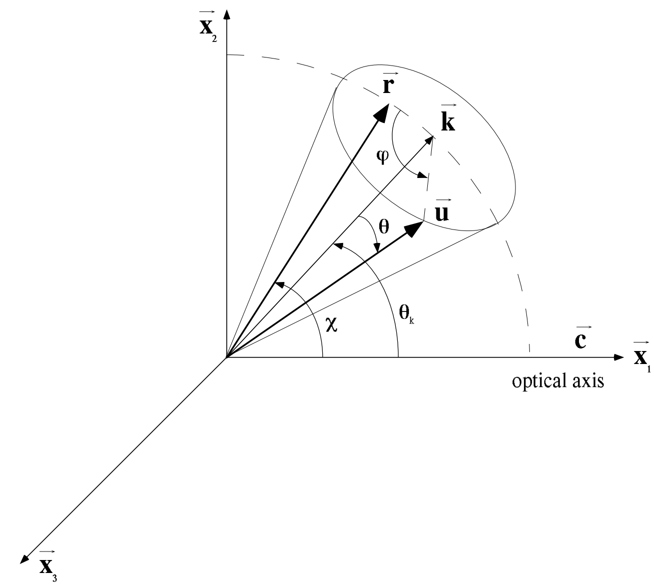

Fig. 1. Coordinate system and main notations.

can be simultaneously emitted, and the total number of Cherenkov photons emitted d^2N by each wave, per path length $\mathrm{d}l$ and per energy $\mathrm{d}E$, is given by the following formula [3]:

$$\frac{\mathrm{d}^2N}{\mathrm{d}l\mathrm{d}E} = \frac{\alpha c^3}{2\pi\hbar\mu}\int_0^{4\pi}\frac{(\mathbf{e}\cdot\mathbf{r})^2}{v_\mathrm{p}^4}\,\delta\!\left(\mathbf{r}\cdot\mathbf{u}-\frac{v_\mathrm{p}}{c\beta}\right)\mathrm{d}\mathbf{u} \qquad (2.1)$$

where:

- α is the fine structure constant;
- μ (= 1 in the following study) is the scalar magnetic permittivity of the medium;
- $\mathrm{d}\mathbf{u}$ is an element of solid angle;
- $\mathbf{u}$ is the unitary vector in the direction of the Cherenkov wave phase propagation ($\mathbf{u}$ is the normal to the wave front and is not necessarily the direction of the light ray);
- $\mathbf{e}$ is a vector in the direction of the electric field;
- v_p is the phase velocity of the emitted wave with:

$$v_\mathrm{p}^2 = \frac{c^2}{\mu}\mathbf{e}\cdot\mathbf{d}; \qquad (2.2)$$

- $\mathbf{d}$ is the unitary vector in the direction of the displacement vector with:

$$\mathbf{u}\cdot\mathbf{d} = 0. \qquad (2.3)$$

By definition:

- the two waves are orthogonally-polarized:

$$\mathbf{d}^{(\mathrm{e})}\cdot\mathbf{d}^{(\mathrm{o})} = 0 \qquad (2.4)$$

- $\mathbf{d}$ and $\mathbf{e}$ are linked by the dielectric tensor:

$$d_i = \sum \epsilon_{ik}e_k. \qquad (2.5)$$

We finally introduce in the $(\mathbf{r},\mathbf{c})$ plane another unitary vector $\mathbf{k}$ from which $\mathbf{u}$ is defined by its polar angle θ and its azimutal angle φ:

$$\begin{aligned} u_1 &= k_1\cos(\theta) - k_2\sin(\theta)\cos(\varphi)\\ u_2 &= k_2\cos(\theta) + k_1\sin(\theta)\cos(\varphi) \qquad (2.6)\\ u_3 &= \sin(\theta)\sin(\varphi). \end{aligned}$$

3 Properties of the ordinary wave

By definition, the displacement vector $\mathbf{d}^{(\mathrm{o})}$ of the ordinary wave is perpendicular to the optical axis $\mathbf{c}$. By using equation (2.3), and $u_2^2+u_3^2 = 1-u_1^2$, we find $\mathbf{d}^{(\mathrm{o})}$ and then $\mathbf{e}^{(\mathrm{o})}$ by equation (2.5):

$$\mathbf{d}^{(\mathrm{o})} = \frac{1}{\sqrt{1-u_1^2}}\left|\begin{matrix}0\\ u_3\\ -u_2\end{matrix}\right. \qquad (3.1)$$

$$\mathbf{e}^{(\mathrm{o})} = \frac{1}{n_\mathrm{o}^2\sqrt{1-u_1^2}}\left|\begin{matrix}0\\ u_3\\ -u_2\end{matrix}\right. \qquad (3.2)$$

$$v_\mathrm{p}^{(\mathrm{o})} = \frac{c}{n_\mathrm{o}}\cdot \qquad (3.3)$$

Since $v_\mathrm{p}^{(\mathrm{o})}$ is constant, by equation (2.1) then shows that the ordinary photons are emitted on a circular cone $\theta_{(\mathrm{o})}$ defined by $\mathbf{r}\cdot\mathbf{u} = \cos(\theta_{(\mathrm{o})})$ with:

$$\cos(\theta_{(\mathrm{o})}) = \frac{1}{n_\mathrm{o}\beta} \qquad (3.4)$$

and the refractive index seen by the ordinary wave is:

$$n_{(\mathrm{o})} = \frac{c}{v_\mathrm{p}^{(\mathrm{o})}} = n_\mathrm{o}. \qquad (3.5)$$

As in an isotropic medium, the symmetry axis of this cone is $\mathbf{r}$ and the particule must move faster than the critical speed $\beta_{(\mathrm{o})}^\mathrm{c}c$ with:

$$\beta_{(\mathrm{o})}^\mathrm{c} = \frac{1}{n_\mathrm{o}}\,\cdot \qquad (3.6)$$

The integration of equation (2.1) over the angle θ then gives:

$$\frac{\mathrm{d}^2N^{(\mathrm{o})}}{\mathrm{d}l\mathrm{d}E} = \frac{\alpha}{2\pi\hbar c}\int_0^{2\pi}\frac{r_2^2u_3^2}{1-u_1^2}\mathrm{d}\varphi \qquad (3.7)$$

where $\mathbf{u}$ is defined by the relations (2.6) in which $\theta = \theta_{(\mathrm{o})}$ and $\mathbf{k} = \mathbf{r}$.

We finally obtain, for an incident particle moving faster than the critical speed, the number of ordinary photons $\mathrm{d}^3N^{(\mathrm{o})}$ emitted per azimutal angle φ, energy $\mathrm{d}E$ and path length $\mathrm{d}l$:

$$\frac{\mathrm{d}^3N^{(\mathrm{o})}}{\mathrm{d}l\mathrm{d}E\mathrm{d}\varphi} = \frac{\alpha}{2\pi\hbar c} \times\frac{\sin^2(\theta_{(\mathrm{o})})\sin^2(\chi)\sin^2(\varphi)}{1-\left(\cos(\theta_{(\mathrm{o})})\cos(\chi)-\sin(\chi)\cos(\chi)\sin(\theta_{(\mathrm{o})})\right)^2}\,. \qquad (3.8)$$

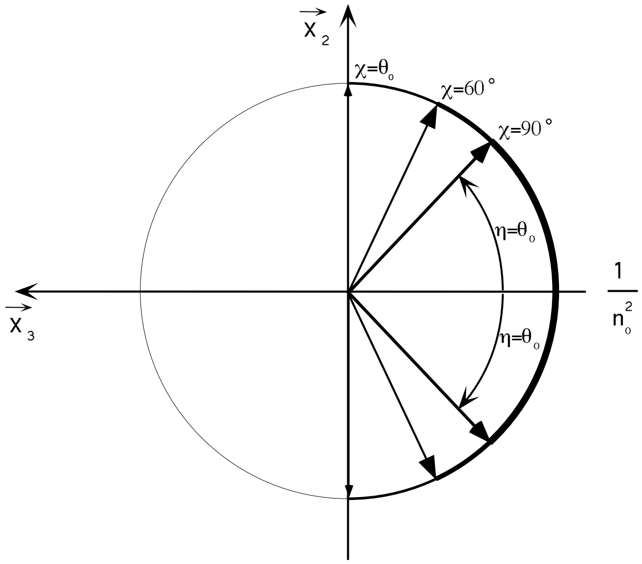

Fig. 2. Ordinary electric field as the normal runs around the cone.

Pafomov [4] and Ginzburg [3] obtained a similar expression in another coordinate system.

We must now calculate the total number of ordinary photons by integrating equation (3.8) over the azimutal angle φ. The result of this integration depends on the sign of $\cos(\chi) - \cos(\theta_{(o)})$ and we get two expressions, one when the optical axis is *inside* the ordinary cone $\left(\chi \leq \theta_{(o)}\right)$ and the other when it is *outside*:

$$\frac{\mathrm{d}^2 N^{(o)}}{\mathrm{d}l\mathrm{d}E} = \frac{\alpha}{\hbar c}\left(1 - \cos(\chi)\right) \qquad \textit{inside} \text{ the cone} \tag{3.9a}$$

$$\frac{\mathrm{d}^2 N^{(o)}}{\mathrm{d}l\mathrm{d}E} = \frac{\alpha}{\hbar c}\left(1 - \cos(\theta_{(o)})\right) \qquad \textit{outside} \text{ the cone.} \tag{3.9b}$$

The polarization features of this wave are quite different from those of the Cherenkov radiation in an isotropic medium for which the electric field lies in the $(\mathbf{r}, \mathbf{u})$ plane. By definition, the polarization of the ordinary wave lies in the plane perpendicular to the optical axis, *i.e.* $(\mathbf{x}_2, \mathbf{x}_3)$, and is of course perpendicular to the photon motion $\mathbf{u}$. This is confirmed by equations (3.1) and (3.2), which show that the electric field $\mathbf{e}$ forms a fan in this plane as the normal runs around the cone (see Fig. 2).

The extreme values are obtained for electric fields located in the two planes tangent to the cone and passing through the optical axis. When the optical axis is inside the cone, these planes do not exist and the polarization vector fills the entire plane and draws a circle of radius $|\mathbf{e}^{(o)}| = \frac{1}{n_o^2}$. When the optical axis goes outside the cone $\left(\chi \geq \theta_{(o)}\right)$, the electric field only draws a fan in this circle, whose vertex angle 2η is found for $|\mathbf{e}^{(o)}|$ minimum:

$$\cos(\eta) = \frac{1}{\sin(\chi)}\sqrt{\cos^2(\theta_{(o)}) - \cos^2(\chi)}. \tag{3.10}$$

The vertex angle of the fan is minimum for a particle moving perpendicularly to the optical axis where it is equal to the ordinary Cherenkov angle $\theta_{(o)}$. Note that $\mathbf{e}^{(o)}$ is along $\mathbf{x}_3$ for a photon emitted in the $(\mathbf{x}_1, \mathbf{x}_2)$ plane ($\varphi = 0°$ and $\varphi = 180°$).

4 Properties of the extraordinary wave

The displacement vector of the extraordinary wave $\mathbf{d}^{(e)}$ is perpendicular to $\mathbf{d}^{(o)}$ (Eq. 2.4). Using equations (2.3) and equation (3.1) we obtain:

$$\mathbf{d}^{(e)} = \frac{1}{\sqrt{1-u_1^2}}\left|\begin{array}{l} 1-u_1^2 \\ -u_1u_2 \\ -u_1u_3 \end{array}\right. \tag{4.1}$$

$$\mathbf{e}^{(e)} = \frac{1}{\sqrt{1-u_1^2}}\left|\begin{array}{l} \dfrac{1-u_1^2}{\epsilon_e} \\ \dfrac{-u_1u_2}{\epsilon_o} \\ \dfrac{-u_1u_3}{\epsilon_o} \end{array}\right. \tag{4.2}$$

$$v_p^{(e)} = c\sqrt{\frac{1}{n_e^2} + \left(\frac{1}{n_o^2} - \frac{1}{n_e^2}\right)u_1^2}. \tag{4.3}$$

The cone of normals of the extraordinary wave is defined by the argument of the Dirac distribution $\mathbf{r}\cdot\mathbf{u} = \frac{v_p^{(e)}}{c\beta}$ in equation (2.1). We write this expression by means of equation (2.6):

$$P\cos^2(\theta) + Q\sin^2(\theta)\cos^2(\varphi) - 2[Ak_1k_2 - r_1r_2(k_1^2 - k_2^2)]\sin(\theta)\cos(\theta)\cos(\varphi) = R \tag{4.4}$$

$$\text{with} \qquad R = \frac{1}{n_e^2\beta^2}$$
$$A = \left(r_1^2 - \frac{1}{n_o^2\beta^2}\right)\left(r_2^2 - \frac{1}{n_e^2\beta^2}\right)$$
$$P = (k_1r_1 + k_2r_2)^2 - \left(\frac{1}{n_o^2} - \frac{1}{n_e^2}\right)\frac{k_1^2}{\beta^2} \tag{4.5}$$
$$Q = (k_2r_1 - k_1r_2)^2 - \left(\frac{1}{n_o^2} - \frac{1}{n_e^2}\right)\frac{k_2^2}{\beta^2}.$$

The interpretation of the cone's shape is simplified by choosing $\mathbf{k}$ to vanish the term $Ak_1k_2 - r_1r_2(k_1^2 - k_2^2)$ in equation (4.4). $\mathbf{k}$ is then the symmetry axis of the cone and, as a unitary vector, is given by:

$$\left|\begin{array}{l} k_1 = \cos(\theta_k) = \dfrac{1}{\sqrt{2}}\sqrt{1 + \dfrac{A}{\sqrt{A^2 + 4r_1^2r_2^2}}} \\ k_2 = \sin(\theta_k) = \dfrac{1}{\sqrt{2}}\sqrt{1 - \dfrac{A}{\sqrt{A^2 + 4r_1^2r_2^2}}}. \end{array}\right. \tag{4.6}$$

The Cherenkov angle $\theta_{(\mathrm{e})}$, which is relative to $\mathbf{k}$ is:

$$\cos(\theta_{(\mathrm{e})}) = \sqrt{\frac{R - Q\cos^2(\varphi)}{P - Q\cos^2(\varphi)}} \tag{4.7}$$

which means that the extraordinary cone has an elliptical shape.

The critical speed is determined by $\theta_{(\mathrm{e})} = 0$ (or $R = P$):

$$\beta^{\mathrm{c}}_{(\mathrm{e})} = \frac{1}{\sqrt{n_{\mathrm{o}}^2\cos^2(\chi) + n_{\mathrm{e}}^2\sin^2(\chi)}}\cdot \tag{4.8}$$

For a particle moving at the critical speed $\beta^{\mathrm{c}}_{(\mathrm{e})}c$, the Cherenkov radiation is emitted along a direction $\mathbf{k}$ different from $\mathbf{r}$ since $\tan(\theta_k) = \left(\frac{n_{\mathrm{e}}}{n_{\mathrm{o}}}\right)^2 \tan(\chi)$. Note that the two cones never intercept, except when the optical axis is tangent to the ordinary cone ($\chi = \theta_{(\mathrm{o})}$) at $\varphi = 180°$: the two cones are then tangent to each other and to the optical axis ($\theta_{(\mathrm{o})} = \theta_k$ for $\varphi = 180°$). The optical axis is then simultaneously inside or outside the two cones. The index of refraction seen by an extraordinary photon now depends on β, χ and φ:

$$n_{(\mathrm{e})} = \frac{c}{v_{\mathrm{p}}^{(\mathrm{e})}} = \left[\frac{1}{n_{\mathrm{e}}^2} + \left(\frac{1}{n_{\mathrm{o}}^2} - \frac{1}{n_{\mathrm{e}}^2}\right)u_1^2\right]^{-\frac{1}{2}}. \tag{4.9}$$

Note that $n_{(\mathrm{e})}$ is equal to n_{e} only in case of an isotropic medium ($n_{\mathrm{o}} = n_{\mathrm{e}}$) or for an extraordinary wave emitted perpendicularly to the optical axis ($u_1 = 0$). Furthermore, in case of a particle moving along the optical axis ($\chi = 0$) the extraordinary cone is circular around the optical axis ($\theta_k = 0$), and the Cherenkov wave has the same geometrical properties as the one emitted in an isotropic medium of refractive index $n^{\parallel}_{(\mathrm{e})}$ independant of φ:

$$\cos(\theta^{\parallel}_{(\mathrm{e})}) = \frac{1}{\beta n^{\parallel}_{(\mathrm{e})}} \tag{4.7 bis}$$

$$n^{\parallel}_{(\mathrm{e})} = n_{\mathrm{e}}\sqrt{1 + \frac{1}{\beta^2}\left(\frac{1}{n_{\mathrm{e}}^2} - \frac{1}{n_{\mathrm{o}}^2}\right)}. \tag{4.9 bis}$$

We now have to determine the number of emitted photons. For that purpose, we use the properties of the Dirac distribution to find:

$$\delta\left(\mathbf{r}\cdot\mathbf{u} - \frac{v_{\mathrm{p}}^{(\mathrm{e})}}{c\beta}\right) = \frac{\mathbf{u}\cdot\mathbf{r}\,\delta\big(\cos(\theta) - \cos(\theta_{(\mathrm{e})})\big)}{\sqrt{(R - Q\cos^2(\varphi))\,(P - Q\cos^2(\varphi))}} \tag{4.10}$$

and we can write the scalar product on the cone of normals as:

$$\mathbf{r}\cdot\mathbf{e}^{(\mathrm{e})} = \frac{\mathbf{u}\cdot\mathbf{r}}{n_{\mathrm{o}}^2\sqrt{1-u_1^2}}\left(r_1\beta^2 n_{\mathrm{o}}^2 \mathbf{u}\cdot\mathbf{r} - u_1\right). \tag{4.11}$$

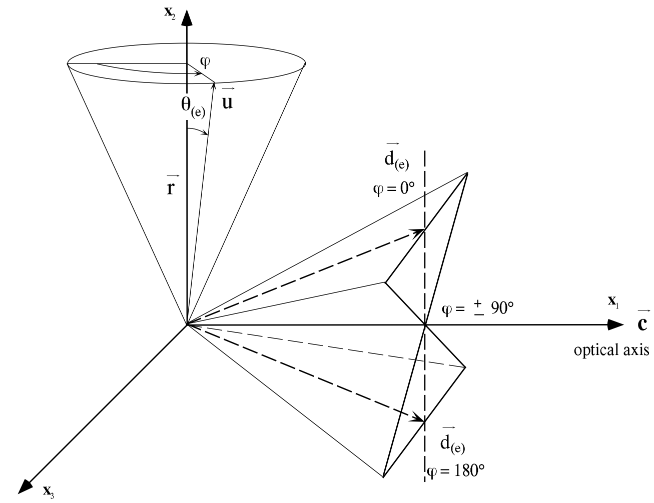

Fig. 3. Extraordinary displacement vector for a particule moving perpendicularly to the optical axis.

This leads, for an incident particle moving faster than the critical speed, to the number of extraordinary photons $\mathrm{d}^3N^{(\mathrm{e})}$ emitted per azimutal angle $\mathrm{d}\varphi$, energy $\mathrm{d}E$ and path length $\mathrm{d}l$:

$$\frac{\mathrm{d}^3N^{(\mathrm{e})}}{\mathrm{d}l\mathrm{d}E\mathrm{d}\varphi} = \frac{\alpha}{2\pi\hbar c}\left(\frac{1}{n_{\mathrm{o}}^2\beta^2}\right) \times \frac{-\dfrac{1}{\mathbf{u}\cdot\mathbf{r}n_{\mathrm{o}}^2\beta^2} + \dfrac{1}{1-u_1^2}\left(\mathbf{u}\cdot\mathbf{r}\left(\dfrac{r_1^2}{n_{\mathrm{o}}^2\beta^2} - 1\right) + 2u_2r_1\right)}{\sqrt{(R - Q\cos^2(\varphi))\,(P - Q\cos^2(\varphi))}}. \tag{4.12}$$

Although this result gives the same number of photons as the one given by Pafomov [4] in the two special cases $r_1 = 1$ ($\mathbf{r} \parallel \mathbf{c}$) and $r_1 = 0$ ($\mathbf{r} \perp \mathbf{c}$), it is different in all the other cases.

We have to integrate equation (4.12) over the azimutal angle φ. After some calculations presented in Appendix A, we find, like in the case of the ordinary wave, that this number depends on the relative position of the optical axis and the cone:

$$\frac{\mathrm{d}^2N^{(\mathrm{e})}}{\mathrm{d}l\mathrm{d}E} = \frac{\alpha}{\hbar c}\left(\cos(\chi) - \frac{\beta^{\mathrm{c}}_{(\mathrm{o})}\beta^{\mathrm{c}}_{(\mathrm{e})}}{\beta^2}\right) \quad \textit{inside the cone} \tag{4.13a}$$

$$\frac{\mathrm{d}^2N^{(\mathrm{e})}}{\mathrm{d}l\mathrm{d}E} = \frac{\alpha}{\hbar c}\left(1 - \frac{\beta^{\mathrm{c}}_{(\mathrm{e})}}{\beta}\right)\frac{\beta^{\mathrm{c}}_{(\mathrm{o})}}{\beta} \quad \textit{outside the cone.} \tag{4.13b}$$

The polarization of the extraordinary wave lies in the plane defined by the photon and the optical axis, *i.e.* $(\mathbf{u}, \mathbf{c})$. The parametric equations of the cone of polarization are rather complicated and Muzicar [7] gives the following estimate:

- If the optical axis is outside the cone, the polarization vector $\mathbf{d}^{(\mathrm{e})}$ is contained in a pyramid whose faces are

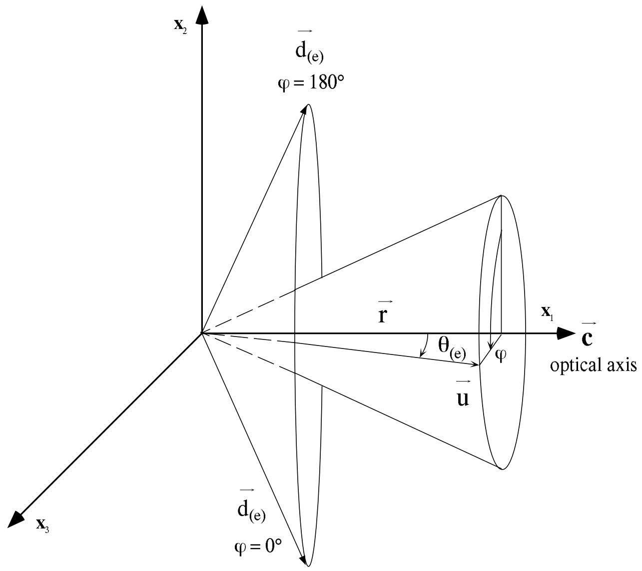

Fig. 4. Extraordinary displacement vector for a particule moving along the optical axis.

the two tangent planes of the cone that pass through the optical axis and the two planes passing through $\mathbf{x}_3$ and perpendicular to the generators of the cone which lie in the plane $(\mathbf{x}_1, \mathbf{x}_2)$ ($\varphi = 0°$ and $\varphi = 180°$) (see Fig. 3).

- If the optical axis is inside the cone, the polarization vector $\mathbf{d}^{(e)}$ generates a cone of same axis $\mathbf{k}$ as the cone of normals (see Fig. 4).

5 Total number of emitted photons

The expression of the total number of photons emitted by a particle depends on the relative value of its speed β with respect to the two critical speeds, the ordinary one $\beta^{c}_{(o)}$ (Eq. (3.6)) and the extraordinary one $\beta^{c}_{(e)}$ (Eq. (4.8)) which depends on the angle χ:

- if β is smaller than the two critical speeds, no photon is emitted.
- if β is greater than the two critical speeds, the two cones are emitted and the total number of emitted photons is the sum of equations (3.9) and (4.13) which gives the same expression whether the optical axis is inside or outside the cone:

$$\frac{\mathrm{d}^2 N^{\mathrm{t}}}{\mathrm{d}l\mathrm{d}E} = \frac{\alpha}{\hbar c}\left(1 - \frac{\beta^{c}_{(o)}\beta^{c}_{(e)}}{\beta^2}\right) \tag{5.1}$$

- if β is between the two critical speeds, only one of the two cones is emitted and the optical axis is always outside the cone. For a negative (positive) material, only the ordinary (extraordinary) cone is emitted and the total number of photons is given by the equations (3.9b) and (4.13b).

For a given angle χ, the total number of emitted photons increases with β. Note that the distribution presents two kinks at the two critical speeds.

But, for a given speed β, the variation of N^{t} with χ depends on the relative value of β with respect to $1/n_{\mathrm{e}}$ and $1/n_{\mathrm{o}}$:

- if $\beta > \frac{1}{n_{\mathrm{o}}}$ and $\beta > \frac{1}{n_{\mathrm{e}}}$, β is greater than the two critical speeds and N^{t} is thus given by equation (5.1). It decreases (increases) with χ for a negative (positive) material from $N^{\mathrm{t}}_{\parallel}$ ($\chi = 0°$) down (up) to $N^{\mathrm{t}}_{\perp}$ ($\chi = 90°$) given by:

$$N^{\mathrm{t}}_{\parallel} = \frac{\alpha}{\hbar c}\left(1 - \frac{1}{n_{\mathrm{o}}^2\beta^2}\right) \tag{5.2a}$$

$$N^{\mathrm{t}}_{\perp} = \frac{\alpha}{\hbar c}\left(1 - \frac{1}{n_{\mathrm{o}}n_{\mathrm{e}}\beta^2}\right). \tag{5.2b}$$

Note that $N^{\mathrm{t}}_{\parallel}$ is equal to the number of photons emitted in an isotropic medium of refractive index n_{o}:

$$N^{\mathrm{t}}_{\mathrm{isot}}(\mathrm{eV}^{-1}\,\mathrm{mm}^{-1}) = 37\sin^2(\theta_{(o)}) \tag{5.3}$$

- for β between $\frac{1}{n_{\mathrm{o}}}$ and $\frac{1}{n_{\mathrm{e}}}$, a particular angle χ_0 occurs when $\beta^{c}_{(e)} = \beta$:

$$\sin(\chi_0) = \sqrt{\frac{1 - \dfrac{1}{n_{\mathrm{o}}^2\beta^2}}{1 - \dfrac{n_{\mathrm{e}}^2}{n_{\mathrm{o}}^2}}}\,. \tag{5.4}$$

For a negative material, ordinary photons are always emitted since $\beta > \beta^{c}_{(o)}$. N^{t} decreases with χ down to χ_0 and is given by equation (5.1) since the two cones are emitted. Then, only ordinary photons are emitted ($\beta < \beta^{c}_{(e)}$) and N^{t} is independant of χ (Eq. (3.9b)).

For a positive material, ordinary photons are never emitted since $\beta < \beta^{c}_{(o)}$. No photons are emitted for $\chi \leq \chi_0$, and then only extraordinary photons are emitted ($\beta > \beta^{c}_{(e)}$) with N^{t} increasing with χ (Eq. (4.13b) since $\chi > \chi_0 > \theta_{(o)}$).

Our results are different from those of Muzicar who made a mistake in the integration of equation (4.12) which has also been misprinted. They are in agreement with the observations of Gföller [6] who found $N^{\mathrm{t}}_{\perp}/N^{\mathrm{t}}_{\parallel} \approx 85\%$ for a ^{32}P β source of 1.71 MeV ($\beta = 0.954$) since equation (5.2) give $N^{\mathrm{t}}_{\perp}/N^{\mathrm{t}}_{\parallel} = 85.2\%$. They are also successfully applied to the isotropic case ($n_{\mathrm{o}} = n_{\mathrm{e}}$) since equation (5.1) reduces, whatever the angle χ, to the well-known formula (5.3) of the classical Cherenkov theory in an isotropic medium.

6 Cherenkov radiation emission in a $NaNO_3$ crystal

As a numerical application, we choose the Cherenkov radiation emission in a highly negative birefringent $NaNO_3$

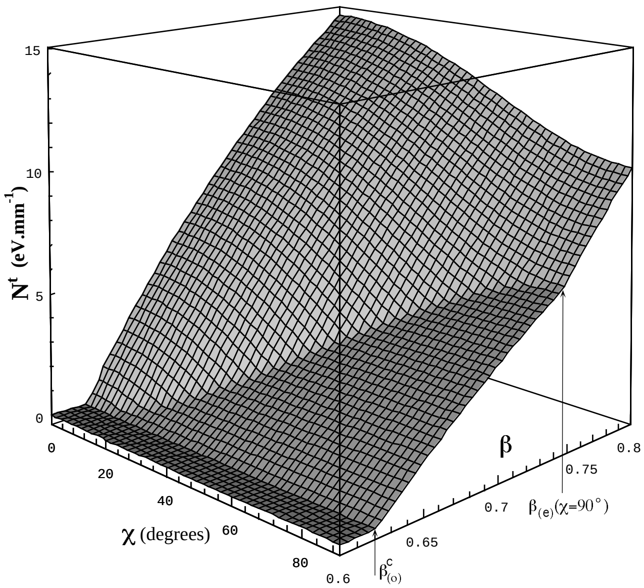

Fig. 5. N^{t} *versus* χ and β in a $\mathrm{NaNO_3}$ crystal.

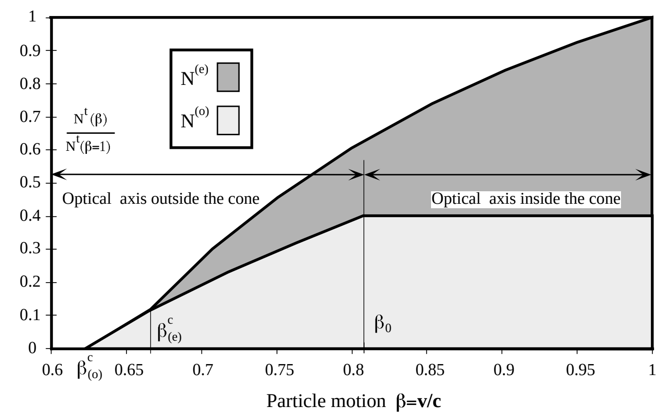

Fig. 6. $N^{(\mathrm{o})}$, $N^{(\mathrm{e})}$ and N^{t} *versus* β in a $\mathrm{NaNO_3}$ crystal at $\chi = 40°$.

crystal whose refractive indices are $n_{\mathrm{o}} = 1.61$ and $n_{\mathrm{e}} = 1.35$ (at $\lambda = 589.3$ nm). Figure 5 shows the dependance of the total number of emitted photons *versus* χ and β. Note the location of the break angle line which runs from the point $(\chi = 0°, \beta = \beta^{\mathrm{c}}_{(\mathrm{o})})$ to the point $(\chi = 90°, \beta = \beta^{\mathrm{c}}_{(\mathrm{e})})$.

The two next figures illustrate the dependance of $N^{(\mathrm{o})}$, $N^{(\mathrm{e})}$, and N^{t} *versus* β. When β increases, the two cones open out and the optical axis can go inside them above the break point speed $\beta_0 = \frac{\beta^{\mathrm{c}}_{(\mathrm{o})}}{\cos(\chi)}$. For $\cos(\chi) > \beta^{\mathrm{c}}_{(\mathrm{o})}$, $\beta_0 < 1$ and we can see the two regimes (Fig. 6 for $\chi = 40°$). For $\cos(\chi) < \beta^{\mathrm{c}}_{(\mathrm{o})}$, $\beta_0 > 1$ and the optical axis is always outside the cones (Fig. 7 for $\chi = 90°$). Note that β_0 is always greater than $\beta^{\mathrm{c}}_{(\mathrm{e})}$ and that it has no effect on the total number of emitted photons N^{t}.

For the dependance of $N^{(\mathrm{o})}$, $N^{(\mathrm{e})}$, and N^{t} *versus* χ, we expect two interesting figures:

- for $\beta > \frac{1}{n_{\mathrm{e}}}$, there is no kink in the N^{t} variation (Fig. 8 for $\beta = 1$);

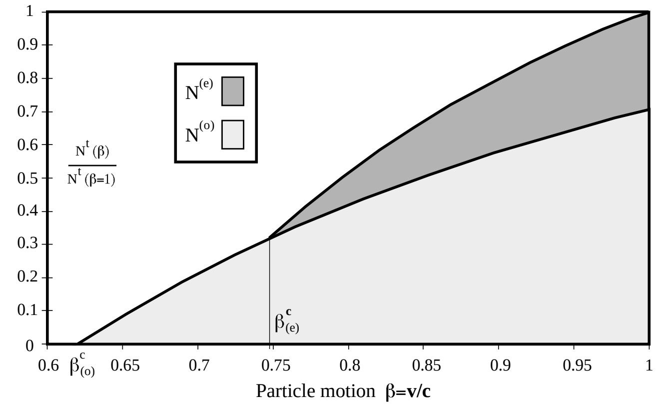

Fig. 7. $N^{(\mathrm{o})}$, $N^{(\mathrm{e})}$ and N^{t} *versus* β in a $\mathrm{NaNO_3}$ crystal at $\chi = 90°$.

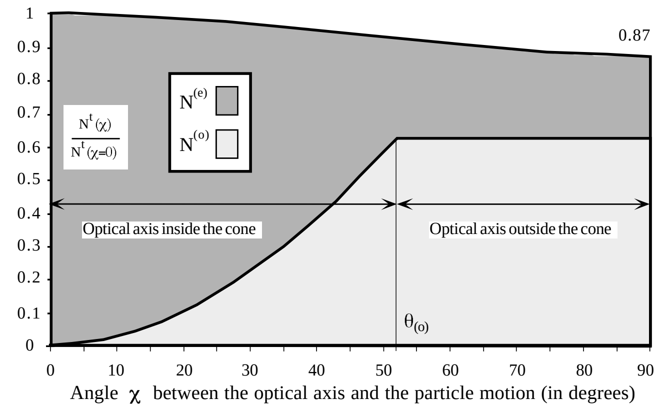

Fig. 8. $N^{(\mathrm{o})}$, $N^{(\mathrm{e})}$ and N^{t} *versus* χ in a $\mathrm{NaNO_3}$ crystal for $\beta = 1$.

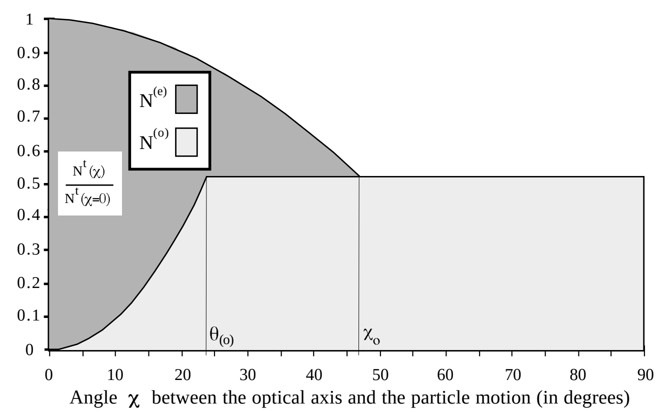

Fig. 9. $N^{(\mathrm{o})}$, $N^{(\mathrm{e})}$ and N^{t} *versus* χ in a $\mathrm{NaNO_3}$ crystal for $\beta = 0.68$.

- for $\frac{1}{n_{\mathrm{o}}} < \beta < \frac{1}{n_{\mathrm{e}}}$, there is a kink at $\chi = \chi_0$ (Fig. 9 for $\beta = 0.68$).

Finally, we used equations (3.8) and (4.12) to study the shape of the cones in the case of a relativistic particle ($\beta = 1$) for $\chi \le \theta_{(\mathrm{o})}$ and $\chi \ge \theta_{(\mathrm{o})}$ (Fig. 10).

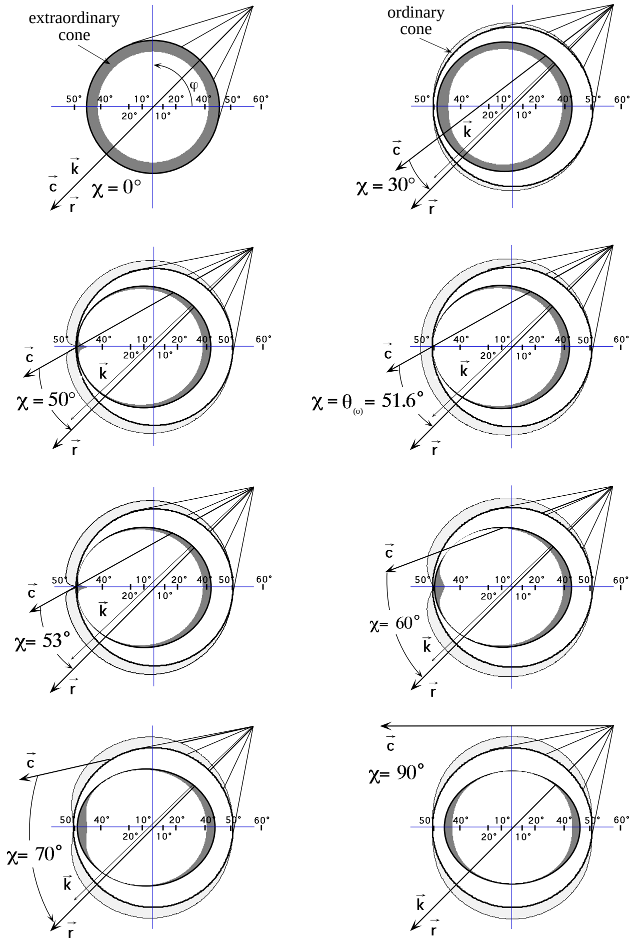

Fig. 10. Ordinary (gray) and extraordinary (black) cone shapes for a relativistic particle moving at various angle χ in a $NaNO_3$ crystal (see details in text).

These figures are cross sections of the two cones in a plane perpendicular to the direction of the particle $\mathbf{r}$:

- the cross section of the ordinary cone for which $\mathbf{r}$ is the symmetry axis, is the same circle whatever the angle χ;
- the cross section of the extraordinary cone is an ellipse whose center is slightly shifted from the symmetry axis $\mathbf{k}$ of this cone, except when the two cones have the same symmetry axis $\mathbf{k} = \mathbf{r}$ for a particle moving along or perpendicularly to the optical axis. In a $NaNO_3$ crystal, the slight difference between the two symmetry axes is not greater than 5 degree.

The energy of each cone is drawn as the thickness of the cross sections: for the ordinary cone, a number proportional to the amount of ordinary emitted photons is added to the circular cross section of the cone, whereas the corresponding number for the extraordinary cone is substracted from its cross section.

The features of the obtained intensities are well explained by polarization considerations. The Cherenkov energy is proportional to the square of the scalar product $\mathbf{e} \cdot \mathbf{r}$ (see Eq. (2.1)). No ordinary wave can thus be emitted for a particle moving along the optical axis and for photons emitted in the $(\mathbf{r}, \mathbf{c})$ plane. A special case occurs when the particle moves at an angle equal to the ordinary Cherenkov angle (Fig. 10 for $\chi = \theta_{(\mathrm{o})} = 51.6°$). The two cones are then tangent to the optical axis and the ordinary photons can be emitted at $\varphi = 180°$ since they are moving along the optical axis.

7 Conclusion

The Cherenkov radiation emission in uniaxial optical materials was extensively studied by Pafomov [4] in 1956 and Muzicar [7] in 1961. These two articles unfortunately lead to different results, especially in the most general case of a particle moving at an arbitrary angle to the optical axis of the material. We have developped a complete formalism and corrected their erroneous conclusions about the energetic properties. This formalism is in agreement with the experiments previously done in the 60's in the two special cases of a particle moving along or perpendicularly to the optical axis of the material: Zrelov [5] for the geometrical properties of the Cherenkov cones and Gföller [6] for their energetic properties. Our results have now to be experimentally verified for particles moving at an arbitrary angle to the optical axis.

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Appendix A

We present here the main stages of the integration of equation (4.12) to derive the total number of extraordinary emitted photons (Eq. (4.13)). It is exactly the way Muzicar did, but his result for I_1 is erroneous (T is false).

After some calculations, we can rewrite equation (4.12) to get:

$$\frac{\mathrm{d}^2 N^{(\mathrm{e})}}{\mathrm{d}l\mathrm{d}E} = \frac{\alpha}{2\pi\hbar c}\left(\frac{1}{n_{\mathrm{o}}^2\beta^2}\right)\{I_1 + I_2\}$$

with:

$$I_1 = -\frac{1}{n_{\mathrm{o}}^2\beta^2}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{\mathbf{u}\cdot\mathbf{r}\sqrt{(R - Q\cos^2(\varphi))\,(P - Q\cos^2(\varphi))}}$$

$$I_2 = \int_0^{2\pi}\frac{\left[\mathbf{u}\cdot\mathbf{r}\left(\frac{r_1^2}{n_{\mathrm{o}}^2\beta^2} - 1\right) + 2u_2 r_1\right]\mathrm{d}\varphi}{(1 - u_1^2)\sqrt{(R - Q\cos^2(\varphi))\,(P - Q\cos^2(\varphi))}}.$$

Using equation (3.4) to express $\mathbf{u} \cdot \mathbf{r}$, and omitting terms odd in $\cos(\varphi)$:

$$I_1 = -\frac{1}{n_o^2\beta^2}\int_0^{2\pi} \frac{k_1r_1 + k_2r_2}{R(k_1r_1 + k_2r_2)^2\sin^2(\varphi) - T\cos^2(\varphi)}\mathrm{d}\varphi$$

with:

$$\begin{aligned} T &= (k_1r_1 + k_2r_2)^2(Q - R) + (k_2r_1 - k_1r_2)^2(P - R) \\ &= -\left(\frac{r_1^2}{n_e^2\beta^2} + \frac{r_2^2}{n_o^2\beta^2}\right) \end{aligned}$$

and therefore:

$$I_1 = -\frac{2\pi n_o}{\sqrt{n_o^2\cos^2(\chi) + n_e^2\sin^2(\chi)}}.$$

For I_2 we get:

$$\begin{aligned} I_2 &= 2\int_{-\infty}^{\infty} \frac{Uy^2 + x}{(Wy^2 - 2iDy - M)(Wy^2 + 2iDy - M)}\mathrm{d}y \\ &= 4\int_{-\infty}^{\infty} \frac{ay + b}{Wy^2 - 2iDy - M}\mathrm{d}y \end{aligned}$$

with:

$$\begin{aligned} W &= P - k_1^2R \\ D &= k_1(P - R) \\ M &= k_1^2(P - R) + k_2^2(Q - R) = r_1^2 - \frac{1}{n_o^2\beta^2} \\ U &= W\left(\frac{M}{n_o^2\beta^2}(k_1r_1 + k_2r_2) + 2k_2r_2\right) \\ x &= \frac{M}{n_o^2\beta^2}\left(2r_1D - (k_1r_1 + k_2r_2)M - 2\frac{k_2r_2}{n_o^2\beta^2}\right) \\ a &= \frac{UM + xW}{4MDi} \\ b &= -\frac{x}{2M}. \end{aligned}$$

The result of the integration is:

$$I_2 = \frac{4\pi i}{W}\frac{1}{z_2 - z_1}\left[(ib - az_1)\mathrm{sign}(z_1) - (ib - az_2)\mathrm{sign}(z_2)\right]$$

where iz_1 and iz_2 are the roots of $(Wy^2 - 2iDy - M)$:

$$z_{1,2} = \frac{D \pm \dfrac{k_2r_2}{n_o\beta}}{W}.$$

z_1 is positive but the sign of z_2 depends on the relative orientation of the optical axis and the ordinary cone of normals. We finally get:

$$\begin{aligned} I_2 &= 2\pi\cos(\chi)n_o^2\beta^2 && \left(\text{for } \cos(\chi) \le \frac{1}{n_o\beta}\right) \\ I_2 &= 2\pi n_o\beta && \left(\text{for } \cos(\chi) \ge \frac{1}{n_o\beta}\right). \end{aligned}$$

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